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AIRFOIL OPTIMIZATION BY THE METHOD OF INVERSE BOUNDARY-VALUE PROBLEMS*

A.M. ELIZAROV and E.V. FEDOROV

Some variational problems on the shape of impermeable wing sections that ensure maximum lift, minimum drag, and maximum aerodynamic quality for a given perimeter and one fixed-angle trailing edge in a non-separating plane steady incompressible viscous flow at high Reynolds numbers are formulated and solved. Functionals are constructed whose minimization is equivalent to the optimization of these characteristics. The existence and uniqueness of extremum points is analysed. Examples of optimized wing sections are given. The variational problems are solved, following /1/, by constructing an operator that acts on the functions of a given set, where each function corresponds to the object required with the necessary properties (in our case, a single-foil section bounded by a closed piecewise-Lyapunov contour).

One of the approaches to airfoil optimization is by solving the direct boundary-value problems of aerodynamics. A multiparameter family of contours of a certain type is defined as alternatives for the modification of the initial contour. The aerodynamic characteristics are calculated for each wing section, and they are optimized by selecting the values of the free parameters subject to constraints that express the conditions of physical and constructive realizability of the mathematical solution (see, e.g., /2, Sect.6/). This approach enables us to allow automatically for changes in the values of side parameters by including them in the system of problem constraints and produces the optimal wing section in a special class, which requires special techniques for modifying the initial contour.

A different approach to airfoil optimization relies on the theory of inverse boundary-value problems (IBVPs) for analytical functions /3-5/, which can be used to solve the problem of constructing wing sections and their grids in an incompressible fluid, in subsonic gas flow, and in a viscous fluid at high Reynolds numbers. The basic IBVP of aerohydrodynamics, which has been studied in maximum detail for the case of an ideal incompressible fluid /3, 6/, involves finding the shape of an impermeable wing section given the flow velocity distribution $v(s)$ along its contour (s is the arc abscissa) for a known unperturbed flow velocity v_∞. If the function $v(s)$ is integrable, this problem is uniquely solvable in the class of Smirnov domains (see, e.g., /7, p.250/) and its solution has an integral representation associated with the conformal mapping of the domain $E^- = \{\zeta: |\zeta| > 1\}$ on the exterior of the required wing section. For various $v(s)$ this representation exhausts the class of Smirnov domains, which is very large and contains domains with rectifiable boundaries (the smoothness of the boundary curves is ensured by additional constraints on $v(s)$).

One of the techniques of airfoil optimization by solving IBVPs involves the optimal choice of $v(s)$. For the design of high-lift airfoils, this approach has been implemented by a number of authors (see the bibliography in /8/) by the optimal choice of the parameters of the initial multiparameter family $v(s)$, constructed taking into account the conditions of hydrodynamic convenience. However, an arbitrary velocity distribution may correspond to a section bounded by a open contour, which represents a physically unrealizable solution. Therefore the basic IBVP of aerohydrodynamics is ill-posed, and its solution in the required class has to be obtained by regularization methods (see /9/), which limits the applicability of this technique.

An alternative technique of optimization by IBVP methods, developed in this paper, relies on Lavrent'ev's idea of specifying the set of feasible solutions as the images of some set of functions under the action of a special operator. This idea has been used /1/ to solve the variational problem of the maximum-lift arc of a given length and bounded curvature in the flow of a smoothly ideal incompressible fluid. In this paper, we apply this idea to develop a set of wing sections bounded by Lyapunov arcs. The corresponding operator is based on the integral representation of the basic IBVP of aerohydrodynamics and the optimizing characteristics are expressed as functionals on the set of functions that satisfy the conditions for the IBVP to be solvable and a general non-separation criterion. The functionals are minimized numerically. Some problems of the existence and uniqueness of the solution of the maximum-

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lift airfoil problem are explored.

Note that the problem of airfoil optimization subject to constraints that prevent flow separation from most of the wing was previously considered in /1/. The numerical solution of this problem /10/ uses a constraint on the pressure gradient as a simplified form of the non-separation condition.

1. ***Statement of the variational problems.*** An impermeable wing section bounded by a closed contour L_z of length L is immersed in an unbounded plane steady incompressible viscous flow at high Reynolds numbers. The flow velocity at infinity is parallel to the abscissa axis in the given coordinate system; its value v_∞ and the fluid density ρ are known.

We start by defining the class of contours L_z. The contours L_z are images of the unit circle under conformal mappings $z = z(\zeta),\ \zeta \in E^-,\ z(\infty) = \infty$, which have the representation

$$z'(\zeta) = (1 - \zeta^{-1})^{\varepsilon-1} z_0'(\zeta),\ \zeta \in E^-;\ 1 \leqslant \varepsilon \leqslant 2,\ z_0'(\zeta) \neq 0$$

and the limiting values

$$\ln |z_0'(e^{i\theta})| = a_0 + p(\theta),\quad a_0 = \text{const}$$

exist. We assume that the function $p(\theta)$ satisfies the Hölder condition $H(K_0, \lambda)$ with constants $0 < K_0 < \infty$ and $0 < \lambda \leqslant 1$, and

$$J_0(p) \equiv \int_0^{2\pi} p(\theta)\, d\theta = 0 \tag{1.1}$$

Taking an arbitrary function $p(\theta)$ that satisfies these conditions, we reconstruct $\ln z'(\zeta)$ by the Schwarz operator with the density $g(\theta) = a_0 + p(\theta) + h(\theta)$, where $h(\theta) = (\varepsilon - 1)\ln[2\sin(\theta/2)]$, and thus obtain

$$z(\zeta) = \int_1^\zeta e^{-i\alpha} \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\, d\theta\right) d\zeta \tag{1.2}$$

where α is a constant that characterizes the attitude of the sections bounded by L_z relative to the incident flow. Taking the boundary-values for $\zeta = e^{i\gamma}$ in (1.2), we obtain a parametric equation of the piecewise-Lyapunov contour L_z which has at most one trailing edge at the point $z = 0$ with a fixed angle $\varepsilon\pi$, $1 \leqslant \varepsilon \leqslant 2$, interior to the flow region. As in /3, 6/, the contour L_z is closed only if

$$J_1(p) + iJ_2(p) \equiv \int_0^{2\pi} p(\theta) e^{i\theta}\, d\theta - \pi(\varepsilon - 1) = 0 \tag{1.3}$$

Therefore, (1.1) and (1.3) are necessary conditions for obtaining L_z from the given class.

Since α is arbitrary, the contours L_z are defined, apart from rotation around the origin. Conversely, for any region bounded by a piecewise-Lyapunov contour of this type (we stress that all contours of this type pass through the point $z = 0$ and have a trailing edge at this point for $\varepsilon > 1$), the conformal mapping $z(\zeta)$ by our choice of normalization $(z(\infty) = \infty$, $z(1) = 0)$ has the form (1.2), and α takes a well-defined value /7/. Taking L_z as given, apart from rotation around the origin, we can find the desired value of α. The operator (1.2) thus describes a large class of sections from the set of sections with piecewise-smooth boundaries.

Note that α in (1.2) has a well-defined physical meaning: it characterizes the deviation of the wing section from the zero-lift flow direction. In what follows we assume that $\alpha > 0$ (this condition corresponds to positive lift) and consider two cases separately, when α is fixed or varies in the range $[0, \pi/2]$. The first case corresponds to a supplementary constraint on the angle of attack of the required section.

It is required to determine the airfoil from a given class of shapes such that, without flow separation, it has highest lift (problem 1), least drag (problem 2), or maximum aerodynamic quality (problem 3).

2. ***Non-separation conditions.*** For steady viscous flow past wing sections at high Reynolds numbers, the typical flow scheme includes a thin boundary layer and a wake. In order to calculate this flow, we consider a model external potential flow past a semi-infinite displacement body, which differs from the original section by its small displacement thickness δ^* and extends into the wake in the form of a thin strip. The velocity distribution on the semibody contour matches the velocity distribution on the wing section contour. Since δ^* is small on the wing section and in the wake, the external flow is approximately identified with

the potential flow of an ideal fluid past the given section. The non-separation condition has the form /11/

$$(-1)^j\,[v(s)]^{-1}v'(s)\delta^{**}(s) \leqslant F(R^{**}),\quad R^{**} = v(s)\delta^{**}(s)/\nu \tag{2.1}$$

ν is the coefficient of kinematic viscosity, $\delta^{**}(s)$ is the momentum loss thickness, $F(R^{**})$ is a function that depends on the flow regime, $j=1$ for the top surface of the wing section, and $j=2$ for the bottom surface. In particular $F(R^{**}) = \mu_1/R^{**}$ or a laminar boundary layer and $F(R^{**}) = \mu_2$ or

$$F(R^{**}) = \mu_3 R^{**-1/m} \tag{2.2}$$

for a turbulent boundary layer, where μ_1, μ_2, μ_3, and m are known empirical constants. From the momentum equation for a turbulent boundary layer (see, e.g., /4, p.398/), we obtain

$$\delta^{**}(s) = A\frac{|v(s)|}{v'(s)}f(s)\,R^{**-1/m},\quad f(s) = a\frac{v'(s)}{|v(s)|^b}\left|\int_{s_*}^{s}|v(\tau)|^{b-1}d\tau\right| \tag{2.3}$$

$$a = (m+1)/m,\quad b = 2(4m+1)/(2m-1)$$

where $A = A(m)$ is a constant that depends on the choice of m /4/, and s_* is the arc abscissa of the flow branching point. Taking the function F in (2.1) in the form (2.2) and using (2.3), we obtain the following non-separation criterion for the turbulent boundary layer (see also /9/):

$$f(s) \geqslant f_*,\quad f_* > f_0 = -\mu_3/A \tag{2.4}$$

where $a = 1.25$, $b = 4.85$, and $f_0 = -5.57\ldots-4.77$ according to Prandtl-Buri, $a = 1.17$, $b = 4.55$, and $f_0 = -3\ldots-2$ according to Loitsyanskii, and $a = 1$, $b = 4$, and $f_0 = -0.8\ldots-0.7$ according to Bam-Zelikovich. Note that (2.4) also gives the non-separation condition by the Kochin-Loitsyanskii method /12, Sect.128/, and $a = 1.17$, $b = 4.75$, $f_0 = -6$ and $a = 0.45$, $b = 5.35$, $f_0 = -0.0681$ for turbulent and laminar bounary layer, respectively.

3. ***Lift maximization. The existence and uniqueness of the solution.*** According to the Zhukovskii-Chaplygin conjecture for $1 < \varepsilon \leqslant 2$, the trailing edge is the point where the flow rolls off the wing section, and the action on the flow on the wing section produces a lift $P = \rho v_\infty \Gamma$, where Γ is the velocity circulation. For $\varepsilon = 1$, the contour is smooth, and in order to determine Γ we will agree that the flow rolls off the wing section at the point $z(1)$. Note that the functions $z(\zeta)$ establish a correspondence between the flows past the required airfoils and the flow past the unit circle. For the circle, the angle of attack is α and the flow rolls at the point $\zeta = 1$.

Let $w(z)$ be the complex flow potentials in the domains D_z that contain ∞ and are bounded by the contours L_z. The functions $w(z)$ are analytical in D_z everywhere except the point at infinity, where they have a simple pole and a logarithmic singularity. In the neighbourhood of ∞,

$$w(z) = v_\infty z + \frac{\Gamma}{2\pi i}\ln z + \sum_{k=0}^{\infty} c_k z^{-k} \tag{3.1}$$

For fixed values of $\rho, v_\infty, \varepsilon$, and α, problem 1 is equivalent to the following variational IBVP: find the region D_z bounded by the contour L_z from the class described above and a function $w(z)$ of the form (3.1) analytical in D_z which satisfies (2.4), where $v(s) = |dw/dz|\,|_{L_z}$, and the conditions

$$\operatorname{Im} w(z)\,|_{L_z} = 0,\quad \Phi \equiv \left|\int_{L_z}(dw/dz)^2\,dz\right| \to \max \tag{3.2}$$

The first relationship in (3.2) is the impermeability condition for L_z, and the second relationship is equivalent to lift maximization, because by Zhukovskii's theorem

$$\Phi = 2v_\infty\Gamma = 2P/\rho = 8\pi v_\infty^2\,|z'(\infty)|\sin\alpha \tag{3.3}$$

Let us reduce the variational IBVP to a variational problem. From the condition defining the perimeter L, we obtain the equality

$$\exp a_0 = 2^{1-\varepsilon}\frac{L}{J(p)},\quad J(p) = \int_0^{2\pi}\sin^{\varepsilon-1}{}^1\!/_2\theta\,\exp p(\theta)\,d\theta \tag{3.4}$$

By (1.2) we have $|z'(\infty)| = \exp a_0$, and therefore by (3.3) and (3.4) maximization of P requires minimization of the functional $J(p)$ on feasible functions from the set

$$U = \{p(\theta) \in H(K_0, \lambda): J_0(p) = 0, \quad J_1(p) + iJ_2(p) = 0\} \tag{3.5}$$

which additionally satisfy (2.4).

Let us investigate the general properties of $J(p)$.

Theorem 1. The functional $J(p)$ is strictly convex on the convex compact set $U \subset L_2[0, 2\pi]$ and the problem of minimizing $J(p)$ on U is therefore uniquely solvable.

Since $J(p) \in C^2(U)$, then for the strict convexity of $J(p)$ it is necessary and sufficient that $\langle(Jp'')\xi, \xi\rangle > 0$ for all $0 \not\equiv \xi \in L_2[0, 2\pi]$, $p \in U$, where $\langle(Jp'')\xi, \xi\rangle$ is the value of the linear functional $(Jp'')(\xi)$ on the element ξ. We have

$$\langle(J''p)\xi, \xi\rangle = \int_0^{2\pi} \sin^{\varepsilon-1}{}^1\!/_2\theta\,\xi^2(\theta)\exp p(\theta)\,d\theta > 0$$

The set U is obviously convex and bounded in the space of Hölder functions. Therefore, U is convex and compact $L_2[0, 2\pi]$.

Consider the problem of minimizing $J(p)$ in the linear subspace $U_0 \subset L_2[0, 2\pi]$ defined by (1.1) and (1.3). Direct application of the Lagrange multiplier method proves the following theorem.

Theorem 2. The functional $J(p)$ attains its global minimum on U_0 at the point $p_*(\theta) = -(\varepsilon - 1)\ln(2\sin{}^1\!/_2\theta)$.

For $\varepsilon = 1$, the function $p_*(\theta) \equiv 0 \in U$, $J(p_*) = 2\pi$, $g(\theta) = \ln[L/(2\pi)]$ and maximum lift is attained for flow past a circle of radius $L/(2\pi)$.

Let $1 < \varepsilon \leqslant 2$. Then $p_*(\theta) \notin U$ and the initial problem is equivalent to the best approximation in $L_2[0, 2\pi]$ of the function p_* by the set U.

Theorem 3. The only best approximation in $L_2[0, 2\pi]$ of the function $p_*(\theta)$ on the set U for some $K_0 = K_0(n)$ is the n-th segment $S_n p_*$ of the Fourier series of the function $p_*(\theta)$ in the trigonometric system $\{\varphi_k\}$. In this case, we have the following bounds which are unimprovable in order of magnitude:

$$\| p_*(\theta) - (S_n p_*)(\theta)\|_{\delta, q} \leqslant C H_q(p_*, \lambda)\, n^{-(\lambda-\delta)}, \quad 1 \geqslant \lambda > \delta > 0 \tag{3.6}$$

$$C = C(p_*, \lambda) = \text{const}, \quad q = 2, \quad \|\cdot\|_{\delta, q} = \|\cdot\|_{L_q} + H_q(\cdot, \delta)$$

$$H_q(p_*, \delta) = \begin{cases} \sup\limits_{t \neq 0} (t^{-\delta}\| p_*(\theta + t) - p_*(\theta)\|_{L_q}), & 0 < \delta < 1 \\ \sup\limits_{t \neq 0} (t^{-1}\| p_*(\theta + t) - 2p_*(\theta) + p_*(\theta - t)\|_{L_q}), & \delta = 1 \end{cases}$$

Proof. Consider the set of trigonometric sums $c_1\varphi_1 + \ldots + c_n\varphi_n$, $c_k = \text{const}$, which is everywhere dense in U. We know (see, e.g., /13/) that $S_n p_*$ is the only best approximation in $L_2[0, 2\pi]$ of the function $p_*(\theta)$ by linear combinations of this kind. Clearly, $S_n p_* \in U$ for some K_0. The deviation $\| S_n p_* - p_*\|_{L_2}$ can be estimated, say, by approximation in H-spaces /14/. In particular, the Banach space $H_q{}^\delta$, $1 \leqslant q < \infty$, $0 < \delta \leqslant 1$, with the norm $\|\cdot\|_{\delta, q}$ consists of functions $r(\theta) \in L_q$ that satisfy the condition $H_q(r, \delta) < \infty$. By the membership criterion in $H_q{}^\delta$ from /14/, $p_*(\theta) \in H_2{}^\lambda$. The bound (3.6) now follows directly from Theorem 3 in /14/.

The function $S_n p_*$ thus solves the problem on the set U for some $K_0 = K_0(n)$; the wing section corresponding to $S_n p_*$ has a trailing edge with the angle $\varepsilon\pi$. From stability theorems for IBVP solutions /5/ it follows that as $n \to \infty$ this wing section approaches a circle without limit. Thus, our results augment and develop the results of /10/.

Additional restrictions can be imposed on U, guaranteeing simplicity and a certain geometrical structure of L_z. These restrictions are expressed in the form of the sufficient single-foil conditions of /15/, which impose constraints on K_0 and restrict U. The set U constructed in this way is treated as the set where the problem is well-posed, and the search for the best approximation to $p_*(\theta)$ on this set is equivalent to finding a quasisolution of the problem. Replacing the function $p_*(\theta)$ by its best approximation $S_n p_*$, we obtain a problem which is equivalent to finding a quasisolution of the external IBVP in U /16/.

Let us now express the criterion (2.4) in terms of the function $p(\theta)$. We can show that condition (2.4) holds everywhere on L_z for $\varepsilon = 2$. In what follows, we consider this specific case.

The complex potential of a flow past a circle has the form

$$w(\zeta) = \Gamma(2\pi)^{-1}[(\zeta e^{-i\alpha} + e^{i\alpha}/\zeta)(2\sin\alpha)^{-1} - i\ln\zeta + \pi + 2\alpha + \operatorname{ctg}\alpha] \tag{3.7}$$

From (3.7) and (3.1) we obtain

$$v[s(\theta)] = \Gamma \cos(^1/_2\theta - \alpha)\exp[-a_0 - p(\theta)]/(2\pi \sin\alpha) \tag{3.8}$$

where $v[s(\theta)] > 0$ for $0 < \theta < \pi + 2\alpha$ (on the top surface of the section), $v[s(\theta)] < 0$ for $\pi + 2\alpha < \theta < 2\pi$ (on the bottom surface). By (3.8), assuming piecewise smoothness of $p(\theta)$, we obtain

$$\frac{d}{d\theta}\ln|v[s(\theta)]| = G_1(p;\theta) \equiv -p'(\theta) - {}^1/_2\,\mathrm{tg}(^1/_2\theta - \alpha)$$

Therefore, the non-separation criterion (2.4) expressed in terms of $p(\theta)$ takes the form

$$(-1)^j G_1(p;\theta) \geqslant f_{0j} G_0(p;\theta),\quad f_{0j} \geqslant f_0,\ j = 1, 2 \tag{3.9}$$

$$\theta \in [0, \pi + 2\alpha] \text{ for } j = 1,\ \theta \in [\pi + 2\alpha, 2\pi] \text{ for } j = 2$$

$$G_0(p;\theta) = G_2(p;\theta)\Big[a\,\Big|\int\limits_{\pi+2\alpha}^{\theta} G_2(p;\theta)\,d\theta\Big|\Big]^{-1}$$

$$G_2(p;\theta) = \sin{}^1/_2\theta\,|\cos(^1/_2\theta - \alpha)|^{b-1}\exp[(2-b)\,p(\theta)]$$

The problem of minimizing the functional $J(p)$ on the set U_1 (U with the supplementary constraints (3.9)) was solved numerically by the relaxation method (see, e.g., /17/).

Fig.1 shows the optimal sections obtained for $\alpha = 0.15 (8.6^\circ)$ and $\alpha = 0.2\ (11.5^\circ)$ (contours 1 and 2), and also the corresponding velocity distributions (curves 3 and 4) (the coordinate system xy was chosen so that the airfoil chord lies on the axis x and the leading edge coincides with the origin). The constants a, b, and f_0 in (2.4) were selected by the Kochin-Loitsyanskii method for a turbulent boundary layer. The dimensionless arc coordinate was related to the perimeter L, the dimensionless velocity $v(s)$ was related to the given value v_∞, and the contour coordinates were related to the airfoil chord c. The section 1 has a lift coefficient $C_y = 1.152$ for angle of attack $\beta = 7.7^\circ$ and relative thickness $t = 0.318$; for section 2, $C_y = 1.502$, $\beta = 10.7^\circ$, and $t = 0.252$.

The constraint (3.9) is very complicated and, in particular, the question of the convexity of U_1 remains open. As simplified non-separation conditions, we can use constraints that define some approximation of the set U_1. The simplest constraints are the inequalities

$$(-1)^j G_1(p;\theta) \geqslant -d_j,\quad j = 1, 2 \tag{3.10}$$

where $d_1, d_2 \geqslant 0$ are constants. Note that for $d_j = 0$ relationships (3.10) give the necessary and sufficient conditions of monotonicity of the function $v(s)$ on the corresponding airfoil surfaces and are therefore exact if we seek a solution with a monotone velocity distribution. The constraints (3.10) conserve the convexity of the feasible set (we denote it by U_2) and guarantee the existence and uniqueness of the minimum point of the functional $J(p)$ on U_2.

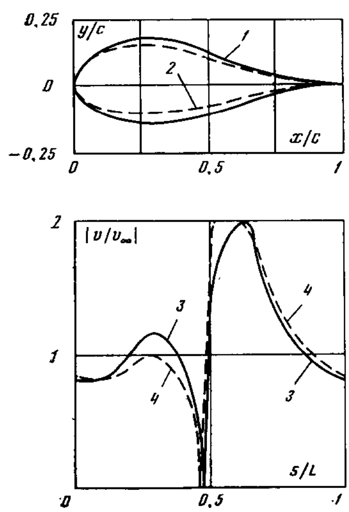

Fig.1

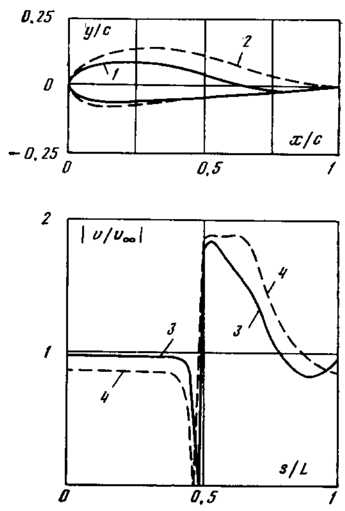

Fig.2

Let us describe the selection of d_1 and d_2. Fixing $d_2 = 0$ and solving the minimization problem for $J(p)$ on the set U_2 for various d_1, we find the maximum possible value $d_{1\max}$ for which the solution of the optimization problem satisfies condition (3.9). For any

$d_1 < d_{1\max}$ we obviously cannot obtain an extremal value of $J(p)$ which is less than that for $d_1 = d_{1\max}$. Let $d_{2,l} = l\eta$, where $l \geqslant 1$ are integers and η is a sufficiently small constant. For each $d_2 = d_{2,l}$ we find the corresponding $d_{1\max}$ by the same technique. We continue increasing d_2 as long as $d_{1\max}$ exists. We thus obtain a set of pairs (d_1, d_2), such that the minimum of $J(p)$ for each pair belongs to the non-empty intersection of U_1 and U_2. The least of the extremum values of $J(p)$ calculated for all pairs is the required solution. A numerical experiment has shown that as d_2 is increased, the corresponding $d_{1\max}$ decreases.

As η is reduced, the proposed algorithm produces the best approximation of the set U_1 by the sets U_2 in the neighbourhood of the extremum point. The implementation of this algorithm involves solving a fairly large number of optimization problems, but at each step the convexity of U_2 and the strict convexity of $J(p)$ guarantee the existence and uniqueness of the extremal function.

Calculations show that this approximation is exact if we seek an airfoil with a monotone velocity distribution on one of the surface (i.e., $f_{0j} = 0$ or $d_j = 0$ for $j = 1$ or 2). The results obtained in the general case and for the approximation (3.10) were virtually identical.

Fig.2 shows the optimal sections with a monotone distribution of $v(s)$ on the bottom surface, constructed for $\alpha = 0.1$ (contour *1*) and $\alpha = 0.2$ (contour *2*) for the same values of a, b, and f_0 as in the previous example. Section *1* has $C_y = 0.696$ for an angle of attack $\beta = 7.6°$ and its relative thickness is $t = 0.157$, for section *2*, $C_y = 1.448$ for $\beta = 9.6°$ and $t = 0.221$. The corresponding $v(s)$ distributions are shown by broken curves *3* and *4*.

Let us consider the case when α is not fixed. By (3.3) and (3.4), lift maximization requires minimizing the functional $J_0(p_0) = J(p)/\sin\alpha$ on the set of feasible solutins $p_0 = (p, \alpha)$, where $p \in U_1$ and $\alpha \in [0, \pi/2]$. A computational experiment has established that the minimum value of $J(p)$ increases as α increases, but much more slowly than $\sin\alpha$. We therefore need to find the maximum α for which the feasible set is non-empty.

Calculations show that as α increases, the optimal wing sections without flow separation become non-single-foil. The maximum α corresponding to a single-foil section for $f_0 = -6$ was found to be 0.28. Fig.3 shows this section and the corresponding $v(s)$ distribution. The wing section has $C_y = 2.017$ for $\beta = 12.8°$ and its relative thickness is $t = 0.252$.

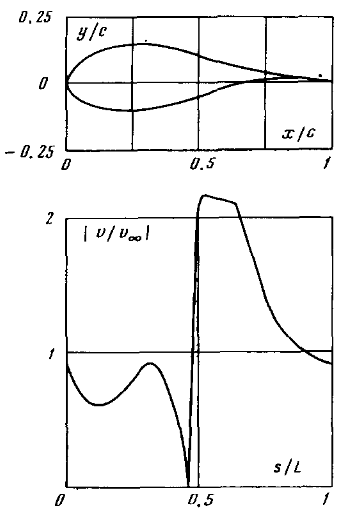

Fig.3

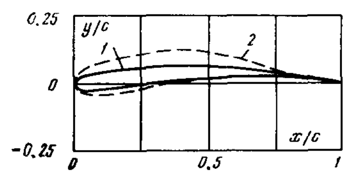

Fig.4

4. *Drag minimization.* In cases without flow separation, the drag coefficient C_x may be calculated from the Squire-Young formula /12, p. 691/

$$C_x = 2\,|v_0/v_\infty|^{3,2}\, c^{-1}\delta_0^{**} \tag{4.1}$$

where $v_0 = v(1)$ and δ_0^{**} is the total momentum loss thickness at the trailing edge. Using relationships (2.3) and (3.8), we obtain from (4.1)

$$C_x = 2^{3,2}(2Aa)^{m/(m+1)}\frac{L}{c}\,\mathrm{Re}^{-1/(m+1)} D(p)$$

$$D(p) = B(p)\,[J(p)]^{-m/(m+1)}, \quad \mathrm{Re} = \frac{v_\infty L}{\nu}$$

$$B(p) = [\cos\alpha\, e^{-p(0)}]^{\mu}\left\{\left[\int_0^{\pi+2\alpha} G_2(p;\theta)\,d\theta\right]^{m/(m+1)} + \left[\int_{\pi+2\alpha}^{2\pi} G_2(p;\theta)\,d\theta\right]^{m/(m+1)}\right\}$$

$$\mu = 3,2 - \frac{6m-1}{2m-1}$$

The problem of drag minimization (Sect.2) is thus equivalent to minimizing the functional $D(p)$ on the set U_1. This problem was solved numerically for a fixed α. Note that the functional $D(p)$ in general is not strictly convex, and therefore the optimization process may produce local minima.

Fig.4 shows single-foil minimum-drag sections constructed for $\alpha = 0.1$ (contour *1*) and $\alpha = 0.2$ (contour *2*) for $\mathrm{Re} = 2 \times 10^6$. Section *1* has $t = 0.07$ and $C_x = 0.0117$, $C_y = 0.655$ for $\beta = 2.46°$; for section *2*, $t = 0.127$, $C_x = 0.0174$ and $C_y = 1.383$ for $\beta = 6.67°$. For comparison, note that for sections *1* and *2* in Fig.2 we have $C_x = 0.0153$ and $C_x = 0.0216$ respectively for the values of β given above.

A computational experiment has shown that C_x decreases as $\alpha \to 0$ for the optimized sections, and the sections themselves become thinner, approaching a plate immersed in a flow at zero angle of attack.

5. ***Maximization of aerodynamic quality.*** From the results of Sections *2* and *4* it follows that maximization of $K = C_y/C_x$ is equivalent to minimization of the functional $E(p) = B(p) [J(p)]^{1/(m+1)} = D(p) J(p)$ on U_1. Calculations show that the behaviour of this functional is very close to that of $D(p)$. This obviously explains why the maximum aerodynamic quality sections obtained for $\alpha = 0.1$ and $\alpha = 0.2$ by solving problem 3 were virtually identical to sections *1* and *2* in Fig.4 (they have $K = 56.9$ and $K = 79.2$, respectively).

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